REGULARITY PROPERTIES OF SOLUTIONS OF SOME ABSTRACT PARABOLIC EQUATIONS†

BY

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ABSTRACT

Consider the equation (i) $(du/dt) - A(t)u(t) = f(t)$ where for $t \in [a, b]$, $A(t)$ is a densely defined and closed linear operator in a Banach space X. Assume the existence of bounded projections $E_i(t)$, $i = 1, 2$, such that $A(t)E_1(t)$ and $-A(t)E_2(t)$ are infinitesimal generators of analytic semigroups and $A(t)$ is completely reduced by the direct sum decomposition $X = \sum_{i=1}^{2} \oplus E_i(t)X$. We show that any solution $u(t)$ of (i) is in $C^\infty(a, b)$ and satisfies the inequalities (1.2) provided that $f(t)$ and $A(t)$ are infinitely differentiable in $[a, b]$ in a suitable sense. In case $A(t)$ and $f(t)$ are in a Gevrey class determined by the constants $\{M_n\}$ we have (1.3). Applications are given to the study of solution of (i) where for $t \in [a, b]$ $A(t)$ is the unbounded operator in $H^{0,p}(G)$ associated with an elliptic boundary value problem that satisfies Agmon's conditions on the rays $\lambda = \pm i\tau$, $\tau > 0$.

1. Introduction

The purpose of this work is to investigate differentiability properties of solutions of the equation

$$(1.1) \qquad \frac{du}{dt} - A(t)u(t) = f(t)$$

where for each $t \in [a, b]$, $A(t)$ is an unbounded operator in $H^{0,p}(G)$ associated with an elliptic boundary value problem. It is assumed that for all sufficiently large real τ and for $\lambda = \pm i\tau$ we have $\lambda \in \rho(A(t))$, the resolvent set of $A(t)$, and $\|(\lambda - A(t))^{-1}\| \leqq C/|\lambda|$.

Earlier results on the differentiability of solutions of the equation (1.1) in a Banach space were obtained by Agmon-Nirenberg [2] for $A(t)$ independent of t.

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For $p = 2$ differentiability properties of solutions of (1.1) follow from the results of [5] and [10]. The proofs in [5] and [10] depend on Hilbert space methods. Differentiability results and Gevrey classes of solutions of initial value problems associated with (1.1) are investigated in [9] assuming that for $t \in [a, b]$, $A(t)$ is the infinitesimal generator of an analytic semi-group. Existence and uniqueness results for weak and strict solutions of a class of two-point problems associated with the equation (1.1) are derived in [4].

In Section 3 we consider the equation (1.1) where for each $t \in [a, b]$, $A(t)$ is a densely defined and closed linear operator in a Banach space X. We assume that there exist bounded projections $E_1(t)$ and $E_2(t)$ in X such that $A(t)E_1(t)$ and $-A(t)E_2(t)$ are infinitesimal generators of analytic semigroups and that $A(t)$ is completely reduced by the direct sum decomposition $X = E_1(t)X \oplus E_2(t)X$. Let $u(t)$ be a solution of (1.1) in $[a, b]$. We prove that $u(t) \in C^\infty(a, b)$ and that for every positive integer n there exists a constant C_n such that for $n = 1, 2, \cdots$ and $t \in (a, b)$ we have

$$
\begin{aligned}
\left\| \frac{d^n}{dt^n} u(t) \right\| \leqq C_n \Big(& \| E_1(a)u(a) \| (t-a)^{-n} + \| E_2(b)u(b) \| (b-t)^{-n} \\
& + \big(\max_{\substack{k=0,\cdots,n \\ t \in [a,b]}} \| f^k(t) \| + \max_{t \in [a,b]} \| u(t) \| \big) \big((t-a)^{-n+1} + (b-t)^{-n+1} \big) \Big)
\end{aligned}
\tag{1.2}
$$

provided that $A(t)$ is infinitely differentiable in $[a, b]$ in a suitable sense and that $f(t) \in C^\infty[a, b]$. In the special case when $A(t)$ and $f(t)$ belong to the Gevrey class $\{M_n\}$ in an appropriate sense we prove the existence of constants C and H such that for $t \in (a, b)$ and $n = 1, 2, \cdots$ we have

$$
\begin{aligned}
\left\| \frac{d^n u}{dt^n} \right\| \leqq CH^n M_n \Big(& (t-a)^{-n} \| E_1(a)u(a) \| + (b-t)^{-n} \| E_2(b)u(b) \| \\
& + (t-a)^{-n+1}(b-t)^{-n+1} \big(\max_{t \in [a,b]} \| u(t) \| + \max_{t \in [a,b]} \| f(t) \| \big) \Big).
\end{aligned}
\tag{1.3}
$$

Applications of the results of Section 3 to the above-mentioned parabolic, boundary value problems are given in Section 4.

2. Notation and definitions

Given two Banach spaces X and Y, we denote by $B(X, Y)$ the space of bounded linear operators from X to Y. The domain of a closed and densely defined linear operator A in X is denoted by $D(A)$. $\rho(A)$ is the resolvent set of A, and $\sigma(A)$ is the spectrum of A. The norm of an element $u \in X$ is denoted by $\| u \|_X$ and, when X is fixed, by $\| u \|$. For $k=0,1,\cdots$ $C^k([a,b],X)$ is the space of k times continuously differentiable functions from the interval $[a,b]$ to X. For $u(t) \in C^k([a,b],X)$ and

$$j = 0,\cdots,k, \ |u(t)|_j = \max_{t \in [a,b]} \left\| \frac{d^j}{dt^j} u(t) \right\|_X.$$

$C([a,b],X) = C^0([a,b],X)$ and $C^\infty([a,b],X) = \bigcap_{k=0}^\infty C^k([a,b],X)$. When X is fixed we set $C^k[a,b] = C^k([a,b],X)$ and $C^\infty[a,b] = C^\infty([a,b],X)$. Denote by $\{M_n\}$ a sequence of positive constants that satisfy the following requirements:

(2.1) $\qquad M_{n+1} \leqq d_0^n M_n$ for all $n \geqq 0$

(2.2) $\qquad \binom{n}{j} M_{n-j} M_j \leqq d_1 M_n$ for all n and j such that $0 \leqq j \leqq n$.

(2.3) $\qquad M_n \leqq M_{n+1}$ for all $n \geqq 0$.

(2.4) $\qquad M_{j+k} \leqq d_2^{j+k} M_j M_k$ for all j and $k \geqq 0$.

d_0, d_1 and d_2 are positive constants. Let $G(H_0, H, [a,b], X)$ be the subset of elements $u(t)$ of $C^\infty([a,b],X)$ that satisfy the inequalities $|u(t)|_n \leqq H_0 H^n N_n$ for $n = 0,1 \cdots$.

We denote by G a bounded domain in R^ν with a boundary ∂G of class C^∞. $\bar{G}$ is the closure of G. $C^\infty(G)$ ($C^\infty(\bar{G})$) is the set of l tuples of infinitely differentiable complex-valued functions that are defined in $G(\bar{G})$. As usual $C_0^\infty(G)$ is the subset of $C^\infty(G)$ consisting of those elements of $C^\infty(G)$ the support of which is a compact subset of G. For $1 < p < \infty$ and $\omega = 0,1,\cdots$, $H^{\omega,p}(G)$ is the completion of $C^\infty(\bar{G})$ under the norm

$$\sum_{|\alpha| \leqq \omega} \left(\int_G \| D^\alpha f(x) \|^p dx \right)^{1/p}.$$

We use the standard notation

$$x = (x_1,\cdots,x_\nu),\ x' = (x_1,\cdots,x_{\nu-1}),\ D_j = i(\partial/\partial x_j),\ D = (D_1,\cdots,D_\nu),$$

and $D^\alpha = D_1^{\alpha_1} \cdots D_\nu^{\alpha_\nu}$. $\alpha = (\alpha_1,\cdots,\alpha_\nu)$ is a multi-index of non-negative integers, $|\alpha| = \sum_{i=1}^\nu \alpha_i$ and for $\zeta \in R^\nu$, $\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_\nu^{\alpha_\nu}$. $R_+^\nu = \{x; x' \in R^{\nu-1}, x_\nu \geqq 0\}$. For $\omega = 0,1,\cdots$ $f \in H^{\omega,p}(G)$ and $L \in B(H^{0,p}(G), H^{\omega,p}(G))$, $\| f \|_\omega$ and $\| L \|_\omega$ denote the norms of f and L as elements of $H^{\omega,p}(G)$ and $B(H^{0,p}(G), H^{\omega,p}(G))$ respectively. $\|f\| = \|f\|_0$ and $\| L \| = \| L \|_0$.

Let $A(x,D)$ be an $l \times l$ system of differential operators that is elliptic of order ω in $\bar{G}$ with coefficients that are infinitely differentiable in $\bar{G}$. Consider boundary operators $B_j(x,D), j=1,\cdots,\frac{1}{2}\omega l$, such that $B_j(x,D)$ is a $1\times l$ system of differential operators of order $\omega_j<\omega$ with coefficients that are infinitely differentiable in $\bar{G}$. Denoted by $H^{\omega,p}(G,\{B_j\})$ the completion of the set $\{u\colon u\in C^\infty(\bar{G}),\ B_j(x,D)u=0$ on ∂G tor $j=1,\cdots,\frac{1}{2}\omega l\}$ in $H^{\omega,p}(G)$. Let A_B^p be the unbounded linear operator in $H^{0,p}(G)$ such that $D(A_B^p)=H^{\omega,p}(G,\{B_j\})$ and $A_B^p u = A(x,D)u$ for $u\in D(A_B^p)$.

For $0\leqq\alpha_1<\alpha_2<2\pi$, set $\Gamma(\alpha_1,\alpha_2)=\{\lambda;\lambda=re^{i\theta}\ r\geqq 0,\alpha_1\ \leqq\theta\leqq\alpha_2\}$ and let $\gamma(\alpha_1,\alpha_2)$ be the boundary of $\Gamma(\alpha_1,\alpha_2)$, that is, positively oriented with respect to $\Gamma(\alpha_1,\alpha_2)$.

Definition 2.1. For $t\in[a,b]$, let $A(t)$ be a closed and densely defined linear operator in a Banach space X. Let $[\alpha,\beta]\subseteq[a,b]$. We say that $u(t)$ is a solution of (1.1) in $[\alpha,\beta]$ if $u(t)\in C[\alpha,\beta]\cap C^1(\alpha,\beta)$; for $t\in(\alpha,\beta)$ we have $u(t)\in D(A(t))$ and $du/dt-A(t)u(t)=f(t)$.

3. Two point problems for ordinary differential equations in a Banach space

For $t\in[a,b]$, let $A(t)$ be a closed and densely defined linear operator in a Banach space X. We assume in Theorem 3.3 below that $A(t)$ satisfies the following conditions.

Condition I. For $i=1,2$ and $t\in[a,b]$, $E_i(t)$ is a bounded projection in X and $A(t)$ is completely reduced by the direct sum decomposition $X=\sum_{i=1}^2\oplus E_i(t)X$.

Condition II. $E_i(t)\in C^\infty([a,b],B(X,X))$ for $i=1,2$.

Condition III. There exist complex numbers μ_i, $i=1,2$, such that for $i=1,2$ the operator $L_i(t)=(-1)^{i+1}(A(t)E_i(t)+\mu_iI)$ satisfies the following three conditions.

(i) For $t\in[a,b]$, the resolvent set of $L_i(t)$ contains the closed sector $\Gamma(-\frac{1}{2}\pi-\theta,\frac{1}{2}\pi+\theta)$ with $0<\theta<\frac{1}{2}\pi$.

(ii) $L_i(t)^{-1}\in C^\infty([a,b],B(X,X))$.

(iii) There exist constants B_n, $n=0,1,\cdots$, for $\lambda\in\Gamma(-\frac{1}{2}\pi-\theta,\frac{1}{2}\pi+\theta)$ and $t\in[a,b]$; we have

$$\left\|\frac{\partial^n}{\partial t^n}(\lambda-L(t))^{-1}\right\|\leqq B_n|\lambda|^{-1}\ . \tag{3.1}$$

Suppose that $L(t)$ satisfies (i), (ii), and (iii). The existence of an evolution operator $U(t,\tau)$ associated with $L(t)$ follows from the results of [6]. It is proved in [9] that $U(t,\tau)$ is infinitely differentiable for $a\leqq\tau<t\leqq b$ and for every pair m,n of non-negative integers there exists a constant $C_{m,n}$ such that

$$\left\| \frac{\partial^n}{\partial t^n} \left(\frac{\partial}{\partial t} \right)^m U(t, \tau) \right\| \leqq C_{m,n} |t - \tau|^{-n}. \tag{3.2}$$

Lemma 3.1 below is a corollary of the proofs in [9].

LEMMA 3.1. *Let $K(t,\tau) \in B(X,X)$ for $a \leqq \tau < t \leqq b$. Assume that for every pair m, n of non-negative integers there exists a constant $C_{m,n}$ such that*

$$\left\| \frac{\partial^n}{\partial t^n} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right)^m K(t, \tau) \right\| \leqq C_{m,n} |t - \tau|^{-n}. \tag{3.3}$$

(i) *Let $f(t) \in C[a,b] \cap C^n(a,b)$. Then for $t \in (a,b)$, $r \in (a,t)$, and $n = 1,2,\cdots$, we have*

$$\begin{aligned} \frac{d^n}{dt^n} \int_a^t K(t,\tau) f(\tau) d\tau &= \int_a^r \frac{\partial^n}{\partial t^n} K(t,\tau) f(\tau) d\tau \\ &+ \sum_{i=0}^{n-1} \sum_{j=0}^{n-1-k} \binom{n-1-k}{j} \left(\frac{\partial}{\partial t} \right)^k \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right)^{n-1-k-j} K(t,r) \frac{d^j}{dt^j} f(r) \\ &+ \int_r^t \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right)^{n-k} K(t,r) \frac{d^k}{d\tau^k} f(\tau) d\tau. \end{aligned} \tag{3.4}$$

(ii) *Let $f(t) \in C^n[a,b]$. Then for $t \in (a,b]$ and $n = 1,2,\cdots$, we have*

$$\begin{aligned} &\frac{d^n}{dt^n} \int_a^t K(t,\tau) f(\tau) d\tau \\ &= \sum_{k=0}^{n-1} \sum_{j=0}^{n-1-k} \binom{n-1-k}{j} \left(\frac{\partial}{\partial t} \right)^k \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right)^{n-1-k-j} K(t,a) \frac{d^j}{d\tau^j} f(a) \\ &+ \int_a^t \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right)^{n-k} K(t,\tau) \frac{d^k}{d\tau^k} f(\tau) d\tau. \end{aligned} \tag{3.5}$$

LEMMA 3.2. *Suppose that $W_1(t,\tau) \in B(X,X)$ for $a \leqq \tau < t \leqq b$, and that $W_2(t,\tau) \in B(X,X)$ for $a \leqq t < \tau \leqq b$. Assume that for $i = 1,2$, $W_i(t,\tau)$ is infinitely differentiable for $t \neq \tau$ and that for every pair m,n of non-negative integers there exists a constant $C_{m,n}$ such that estimate (3.2) holds for $U(t,\tau) = W_i(t,\tau)$.*

Let $[\alpha,\beta] \subseteq [a,b]$. Let n be a positive integer. Assume that $C_{0,0}(\beta-\alpha) < 1$ and, that $2^{k-1}C_{0,0}(\beta-\alpha) < 1$ for $k = 1,\cdots,n$. Let $g_i(t) \in C[\alpha,\beta] \cap C^n(\alpha,\beta)$ for $i = 1,2\cdots$. Suppose that $v_i(t) \in C[\alpha,\beta]$ for $i = 1,2$ and satisfies the relations

$$v_1(t) = g_1(t) + \int_\alpha^t W_1(t,\tau) v_2(\tau) d\tau \text{ and } v_2(t) = g_2(t) + \int_t^\beta W_2(t,\tau) v_1(\tau) d\tau.$$

(i) $v_i(t) \in C^n(\alpha, \beta)$ for $i = 1, 2$, and there exists a constant K_n such that the estimate

$$\left\| \frac{d^n}{dt^n} v_i(t) \right\| \leqq K_n G((t-\alpha)^{-n+1} + (\beta - t)^{-n+1}) \tag{3.6}$$

holds for $i = 1, 2$ and $t \in (\alpha, \beta)$ provided that for $i = 1, 2$, $t \in (\alpha, \beta)$, and $j = 1, \cdots, n$ we have

$$\| g_i(t) \| \leqq G \text{ and} \tag{3.7}$$

$$\left\| \frac{d^j}{dt^j} g_i(t) \right\| \leqq G((t-\alpha)^{-j+1} + (\beta - t)^{-j+1}). \tag{3.8}$$

(ii) $v_i(t) \in C^n(\alpha, \beta)$ for $i = 1, 2$, and there exists a constant K_n such that the estimates

$$\left\| \frac{d^n}{dt^n} v_i(t) \right\| \leqq K_n G((t-\alpha)^{-n} + (\beta - t)^{-n+1}) \text{ and} \tag{3.9}$$

$$\left\| \frac{d^n}{dt^n} v_2(t) \right\| \leqq K_n G((t-\alpha)^{-n+1} + (\beta - t)^{-n+1}) \tag{3.10}$$

hold for $t \in (\alpha, \beta)$ provided that $g_2(t) \equiv 0$ in $[\alpha, \beta]$, that $g_1(t) = h(t)$ and for $j = 0, \cdots, n$ and $t \in (\alpha, \beta)$ we have

$$\left\| \frac{d^j}{dt^j} h(t) \right\| \leqq G(t-\alpha)^{-j}. \tag{3.11}$$

Proof. Let Q_1, Q_2 be the bounded operators from $C[\alpha, \beta]$ to $C[\alpha, \beta]$ that are defined by

$$Q_1 g(t) = \int_\alpha^t W_1(t, \tau) g(\tau) d\tau \text{ and} \tag{3.12}$$

$$Q_2 g(t) = \int_t^\beta W_2(t, \tau) g(\tau) d\tau \tag{3.13}$$

respectively. Let $R_1 = Q_2 Q_1$ and set $R_2 = Q_1 Q_2$.

Let $f(t) \in C[\alpha, \beta] \cap C^n(\alpha, \beta)$ and suppose that for $j = 1, \cdots, n$ and $t \in (\alpha, \beta)$ we have

$$\| f(t) \| \leqq F_0 \text{ and} \tag{3.14}$$

$$\left\| \frac{d^j}{dt^j} f(t) \right\| \leqq F_j((t-\alpha)^{-j+1} + (\beta - t)^{-j+1}). \tag{3.15}$$

For $j = 0, \cdots, n$ let $F'_j = \max_{i=0,\dots,j} F_i$. For $1 \leqq k \leqq n$ there exists a constant C_k such that for $i = 1,2$

$$(3.16)\qquad \left\| \frac{d^k}{dt^k} Q_i f(t) \right\| \leqq (C_k F'_{k-1} + 2^{k-1} C_{0,0}(\beta - \alpha) F_k)((t-\alpha)^{-k+1} + (\beta - t)^{-k+1})$$

provided that $f(t)$ satisfies (3.14) and (3.15). To verify the last assertion for $i = 1$, observe that $(d^k/dt^k)Q_1 f(t)$ is given by the right-hand side of (3.4) with $n = k$, $K(t,\tau) = W_1(t,\tau)$, and $r = \frac{1}{2}(\alpha + t)$. Estimate (3.16) for $i = 2$ is obtained similarly. Consequently for $k = 1, \cdots, n$ there exists a constant C_k such that

$$(3.17)\qquad \left\| \frac{d^k}{dt^k} R_i f(t) \right\| \leqq (C_k F'_{k-1} + (2^{k-1} C_{0\,0}(\beta - \alpha))^2 F_k)((t-\alpha)^{-k+1} + (\beta - t)^{-k+1})$$

for $i = 1,2$ and for $t \in (\alpha, \beta)$. Hence for $k = 1, \cdots, n$ and $j = 0, 1 \cdots$ there exists a constant $F_{k,j}$ such that

$$\left\| \frac{d^k}{dt^k} R_1^j f(t) \right\| \leqq F_{k,j}((t-\alpha)^{-k+1} + (\beta - t)^{-k+1}).$$

Set $G_{0,j} = \left| R_1^j f(t) \right|_0$ and

$$G_{k,j} = \sup_{t \in (\alpha,\beta)} ((t-\alpha)^{-k+1} + (\beta - t)^{-k+1})^{-1} \left\| \frac{d^k}{dt^k} R_1^j f(t) \right\|.$$

Suppose that $C_{0,0}(\beta - \alpha) \leqq \rho < 1$ and that for $k = 1, \cdots, n$, $2^{k-1} C_{0,0}(\beta - \alpha) \leqq \rho < 1$. Then $\sum_{j=0}^{\infty} G_{0,j}$ converges and

$$(3.18)\qquad \sum_{j=0}^{\infty} G_{0,j} \leqq (1 - \rho^2)^{-1} G_{0,0}.$$

It also follows from (3.17) that for $k = 1, \cdots, n$, $\sum_{j=0}^{\infty} G_{k,j}$ converges and that

$$(3.19)\qquad \sum_{j=0}^{\infty} G_{k,j} \leqq (1 - \rho^2)^{-1} \left(C_k \sum_{j=0}^{\infty} G'_{k-1,j} + G_{k,0} \right).$$

Here

$$G'_{k-1,j} = \max_{0 \leqq i \leqq k-1} G_{i,j}.$$

This implies that

$$\sum_{j=0}^{\infty} R_1^j f(t) \in C[\alpha,\beta] \cap C^n(\alpha,\beta)$$

and that there exists a constant K_n such that for $i = 1$ and $t \in (\alpha, \beta)$ we have

$$(3.20)\qquad \left\| \frac{d^n}{dt^n} \sum_{j=0}^{\infty} R_i^j f(t) \right\| \leqq K_n F'_n((t-\alpha)^{-n+1} + (\beta - t)^{-n+1}).$$

The same result holds also for $i = 2$. To complete the proof of (i) it is now sufficient to observe that since $C_{0\,0}(\beta - \alpha) < 1$, $v_1(t)$ and $v_2(t)$ are given by

$$v_1(t) = \sum_{j=0}^{\infty} R_2^j g_1(t) + Q_1 \sum_{j=0}^{\infty} R_1^j g_2(t) \text{ and} \tag{3.21}$$

$$v_2(t) = Q_2 \sum_{j=0}^{\infty} R_2^j g_1(t) + \sum_{j=0}^{\infty} R_1^j g_2(t). \tag{3.22}$$

To prove (ii) one verifies, with the help of Lemma 3.1 (i) and (ii), the existence of constants k_i, $i = 1,\cdots,n$ and C_j, $j = 1,\cdots,n$ such that

$$\left\|\frac{d}{dt} Q_2 h(t)\right\| \leqq K_1 G \ln(t - \alpha) \tag{3.23}$$

and in case $n > 1$,

$$\left\|\frac{d^i}{dt^i} Q_2 h(t)\right\| \leqq K_i G((t-\alpha)^{-i+1} + (\beta - t)^{-i+1}) \tag{3.24}$$

for $i = 2,\cdots,n$ and for $j = 1,\cdots,n$

$$\left\|\frac{d^j}{dt^j} R_2 h(t)\right\| \leqq C_j G((t-\alpha)^{-j+1} + (\beta - t)^{-j+1}). \tag{3.25}$$

Let $u_1(t)$, $u_2(t)$ be the solution of the system of equations

$$u_1(t) = g_1(t) + Q_1 u_2(t) \text{ and} \tag{3.26}$$

$$u_2(t) = g_2(t) + Q_2 u_1(t) \tag{3.27}$$

with $g_1(t) = R_2 h(t)$ and $g_2(t) \equiv 0$. Then $v_1(t) = u_1(t) + h(t)$, $v_2(t) = u_2(t) + Q_2 h(t)$, and the assertion of (ii) follows from (i) and from estimates (3.23) and (3.24).

Assume that $A(t)$ satisfies conditions I, II, and III and for $i = 1, 2$, set $B_i(t) = \sum_{j=1}^{2} E_j'(t)E_j(t) - \mu_i I$ and let $L_i(t) = (-1)^{i+1}(A(t)E_i(t) + \mu_i I)$. As in [4] let $K_1(t,\tau)$, for $a \leqq \tau \leqq t \leqq b$, be the evolution operator associated with $L_1(t) + B_1(t)$. Let $H(t,\tau)$, for $a \leqq \tau \leqq t \leqq b$, be the evolution operator associated with $L_2(a + b - t) - B_2(a + b - t)$, and for $a \leqq t \leqq \tau \leqq b$, set $K_2(t,\tau) = H(a + b - t, a + b - \tau)$. Define $W_1(t,\tau)$ for $a \leqq \tau \leqq t \leqq b$, $W_2(t,\tau)$ for $a \leqq \tau \leqq t \leqq b$, and $W_2(t,\tau)$ for $a \leqq t \leqq \tau \leqq b$ by

$$W_i(t,\tau) = K_i(t,\tau)E_i(\tau)E_i'(\tau). \tag{3.28}$$

It follows from [4, Lem. 3.3] that if $u(t)$ is a solution of (1.1) in $[\alpha,\beta] \subseteq [a,b]$ and, for $i = 1, 2$, $u_i(t) = E_i(t)u(t)$, then for $t \in [\alpha,\beta]$

$$(3.29)\qquad u_1(t) = K_1(t,\alpha)u_\alpha + \int_\alpha^t K_1(t,\tau)E_1(\tau)f(\tau)d\tau + \int_\alpha^t W_1(t,\tau)u_2(\tau)d\tau$$

$$(3.30)\qquad u_2(t) = K_2(t,\beta)u_\beta + \int_t^\beta K_2(t,\tau)E_2(\tau)f(\tau)d\tau + \int_t^\beta W_2(t,\tau)u_1(\tau)d\tau.$$

THEOREM 3.3. *Suppose that $A(t)$ satisfies Conditions I, II, and III. Then every solution of* (1.1) *in* $[a,b]$ *is in* $C^\infty(a,b)$ *provided that* $f(t)\in C^\infty[a,b]$. *For* $n=1,2,\cdots$ *there exists a constant* C_n *such that if* $u(t)$ *is a solution of* (1.1) *in* $[a,b]$ *and* $f(t)\in C^\infty[a,b]$ *then*

$$(3.31)\qquad \left\|\frac{d^n}{dt^n}u(t)\right\| \leqq C_n(\|E_1(a)u(a)\|(t-a)^{-n} + \|E_2(b)u(b)\|(b-t)^{-n} + (|f(t)|_n + |u(t)|_0)((t-a)^{-n+1} + (b-t)^{-n+1}))$$

for $t\in(a,b)$.

PROOF. The assumptions of the present theorem and the relation

$$(\lambda - (L_1(t) + B(t)))^{-1} = (I - (\lambda - L_1(t))^{-1}B(t))^{-1}(\lambda - L_1(t))^{-1}$$

that holds for $\lambda\in\Gamma(-\tfrac{1}{2}\pi - \theta; \tfrac{1}{2}\pi + \theta)$ with

$$|\lambda| \geqq 2B_0 \max_{t\in[a,b]} |B(t)|$$

imply that there exists a complex μ such that $L(t) = L_1(t) + B(t) + \mu I$ satisfies (i), (ii), and (iii) of Condition III. Let $U(t,\tau)$ be the evolution operator associated with $L(t)$. Then $K_1(t,\tau) = e^{-\mu(t-\tau)}U(t,\tau)$. The above-mentioned results of [9] and the assumptions of the present theorem guarantee that for every pair m,n of non-negative integers and for $i = 1$ there exist constants $B_{m,n}$ and $C_{m,n}$ such that

$$(3.32)\qquad \left\|\frac{\partial^n}{\partial t^n}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial\tau}\right)^m K_i(t,\tau)\right\| \leqq B_{m,n}|t-\tau|^{-n} \text{ and}$$

$$(3.33)\qquad \left\|\frac{\partial^n}{\partial t^n}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial\tau}\right)^m W_i(t,\tau)\right\| \leqq C_{m,n}|t-\tau|^{-n}.$$

The same result holds also for $i = 2$.

Let $u(t)$ be a solution of (1.1) in $[a,b]$. For $i = 1,2$ set $u_i(t) = E_i(t)u(t)$. Let $[\alpha,\beta]\subseteq[a,b]$. Let $0<\rho<1$; suppose that $C_{0,0}(\beta-\alpha)\leqq\rho$ and that $2^{k-1}C_{0,0}(\beta,\alpha)\leqq\rho$ for $k = 1,\cdots,n$. Let $v_1^1(t)$, $v_2^1(t)$ be the solution of (3.34) and (3.35) with $g_1(t) = \int_\alpha^t K_1(t,\tau)E_1(\tau)f(\tau)d\tau$ and $g_2(t) = \int_t^\beta K_2(t,\tau)E_2(\tau)f(\tau)d\tau$. Observe that by Lemma 3.1 (ii) the functions $g_i(t)$ satisfy the requirements of Lemma 3.2 (i). Let

$v_1^2(t)$, $v_2^2(t)$ be the solution of (3.26) and (3.27) with $g_1(t) = K_1(t,\alpha)E_1(\alpha)\mu(\alpha)$ and $g_2(t) \equiv 0$; denote by $v_1^3(t)$, $v_2^3(t)$ the solution of (3.26) and (3.27) with $g_1(t) \equiv 0$ and $g_2(t) = K_2(t,\beta)E_2(\beta)u(\beta)$. By [4, Lem. 3.3] $u_i(t)$, $i = 1,2$, satisfy the relations (3.29) and (3.30) and, since $C_{0,0}(\beta-\alpha) < 1$, we have $u_i(t) = \sum_{j=1}^3 v_i^j(t)$, $i = 1,2$. Hence Lemma 3.2 (i) and (ii) guarantee that $u_i(t) \in C^n(\alpha,\beta)$ and that for $t \in (\alpha,\beta)$

$$(3.34)\quad \left\| \frac{d^n}{dt^n} u(t) \right\| \leqq C_n(\| E_1(\alpha)u(\alpha) \| (t-\alpha)^{-n} + \| E_2(\beta)u(\beta) \| (\beta - t)^{-n} + |f(t)|_n).$$

Consequently for every non-negative integer n, $u(t) \in C^n(a,b)$ and there exists a constant C_n such that (3.31) is satisfied.

The following lemma is a consequence of the results of [9].

LEMMA 3.4. *Let $K(t,\tau) \in B(X,X)$ for $a \leqq \tau < t \leqq b$. Suppose that there exist constants N_0 and N such that for every pair m,n of non-negative integers we have*

$$(3.35)\quad \left\| \frac{\partial^n}{\partial t^n} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right)^m K(t,\tau) \right\| \leqq N_0 N^{m+n} M_n M_m (t-\tau)^{-n}.$$

(i) *Let $[\alpha,\beta] \subseteq [a,b]$. For $t \in (\alpha,\beta)$ let $r_n(t) = \alpha + n(t-\alpha)/(n+1)$. There exists a constant C such that for every positive integer n and $t \in (\alpha,\beta)$ we have*

$$(3.36)\quad \begin{aligned} &\left\| \frac{d^n}{dt^n} \int_\alpha^t K(t,\tau) g(\tau) d\tau \right\| \\ &\leqq C G_0 G^{n-1} (t-\alpha)^{-n+1} M_{n-1} + \int_{r_n(t)}^t \left\| \frac{d^n}{dt^n} g(\tau) \right\| d\tau \end{aligned}$$

provided that $g(t) \in C[\alpha,\beta] \cap C^n(\alpha,\beta)$, that for $j = 0,\cdots,n-1$ we have

$$(3.37)\quad \left\| \frac{d^j}{dt^j} g(t) \right\| \leqq G_0 G^j M_j (t-\alpha)^{-j}$$

and that $G > \max(2N, 2N(b-a))$.

(ii) *Let $f \in C^\infty[a,b]$ and assume that there exist constants F_0 and F such that for every non-negative integer n and $t \in [a,b]$ we have*

$$(3.38)\quad \left\| \frac{d^n}{dt^n} f(t) \right\| \leqq F_0 F^n M_n.$$

Then for every positive integer n

$$(3.39)\quad \left\| \frac{d^n}{dt^n} \int_a^t K(t,\tau) f(\tau)\, d\tau \right\| \leqq \tilde{F}_0 \tilde{F} M_n (t-a)^{-n+1}.$$

Here $\tilde{F}_0 = 6d_1 N_0 F_0$ and $\tilde{F} = \max(1, F(b-a), 2N(b-a), 2N)$. The proof of (i) is part of the proof of [9, Th. 3.1]. The proof of (ii) is part of the proof of [9, Th. 3.3].

LEMMA 3.5. *Let $W_1(t,\tau) \in B(X,X)$ for $a \leqq \tau < t \leqq b$ and let $W_2(t,\tau) \in B(X,X)$ for $a \leqq t \leqq \tau \leqq b$. Suppose that for $i = 1,2$, $W_i(t,\tau)$ is infinitely differentiable for $t \neq \tau$. Assume that there exist constants N_0 and N such that for every pair m,n of non-negative integers and for $i = 1,2$ we have*

$$\left\| \frac{\partial^n}{\partial t^n} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right)^m W_i(t,\tau) \right\| \leqq N_0 N^{m+n} M_m M_n (t-\tau)^{-n}. \tag{3.40}$$

Suppose that $N_0 M_0^2(\beta - \alpha) < \frac{1}{2}$.

(i) *Let $g_i(t) \in C[\alpha,\beta] \cap C^\infty(\alpha,\beta)$ for $i = 1,2$. Suppose that there exist constants G_0 and G such that for $i = 1,2$ and for every positive integer n we have*

$$\| g_i(t) \| \leqq G_0 M_0, \tag{3.41}$$

$$\left\| \frac{d^n}{dt^n} g_1(t) \right\| \leqq G_0 G^n M_n (t-\alpha)^{-n+1}, \text{ and} \tag{3.42}$$

$$\left\| \frac{d^n}{dt^n} g_2(t) \right\| \leqq G_0 G^n M_n (\beta - t)^{-n+1}. \tag{3.43}$$

Let $u_1(t)$, $u_2(t)$ be the solution of (3.26) and (3.27). Suppose that the constants C_1 and L satisfy the following conditions: $C_1(\beta-\alpha)^2 \geqq 4$, $C_1(\beta-\alpha) \geqq 4$, $L \geqq G(\beta-\alpha)$, and $L \geqq 2C(\beta-\alpha)$ where C is the constant appearing in the right-hand side of (3.36). Then for every positive integer n and for $i = 1,2$ we have

$$\| u_i(t) \| \leqq 2G_0 M_0 \text{ and} \tag{3.44}$$

$$\left\| \frac{d^n}{dt^n} u_i(t) \right\| \leqq C_1 G_0 L^n M_n (t-\alpha)^{-n+1} (\beta - t)^{-n+1}. \tag{3.45}$$

(ii) *Let $g_1(t) \in C[\alpha,\beta] \cap C^\infty(\alpha,\beta)$ and suppose that there exist constants G_0 and G such that for every non-negative integer n and for $t \in (\alpha,\beta)$ we have*

$$\left\| \frac{d^n}{dt^n} g_1(t) \right\| \leqq G_0 G^n M_n (t-\alpha)^{-n}. \tag{3.46}$$

Let $u_1(t)$, $u_2(t)$ be the solution of (3.26) and (3.27) with $g_2(t) \equiv 0$ and $g_1(t)$ as above. Suppose that the constants C_1 and L satisfy the following conditions $C_1(\beta-\alpha) \geqq 4$, $L \geqq G(\beta-\alpha)$, and $L \geqq 4C(\beta-\alpha)^2$ where C is the constant appearing in the right hand side of (3.36). Then for $i = 1,2$ and for every positive integer n we have

$$\| u_i(t) \| \leqq 2G_0 M_0 \text{ and} \tag{3.47}$$

$$\left\| \frac{d^n}{dt^n} u_i(t) \right\| \leqq C_1 G_0 L^n M_n (t-\alpha)^{-n} (\beta - t)^{-n+1}. \tag{3.48}$$

PROOF. Estimate (3.44) follows from (3.41) and from the assumption that $N_0 M_0^2(\beta-\alpha) < \frac{1}{2}$. Let n be a positive integer. In case $n > 1$ assume that estimate (3.45) holds for $j = 1, \cdots, n-1$. Then

$$\left\| \frac{d^j}{dt^j} u_1(t) \right\| \leqq \frac{C_1(\beta-\alpha)^2}{2} G_0 L^j M_j (t-\alpha)^{-j} (\beta - t)^{-j} \tag{3.49}$$

for $j = 0, \cdots, n-1$ and it follows from Lemma 3.4 (i) that

$$\begin{aligned} \left\| \frac{d^n}{dt^n} u_1(t) \right\| &\leqq G_0 G^n M_n (t-\alpha)^{-n+1} + \frac{CC_1 G_0(\beta-\alpha)^2}{2} \times \\ &\times L^{n-1}(\beta-t)^{-n+1}(t-\alpha)^{-n+1} M_{n-1} + N_0 M_0^2 \int_{r_n(t)}^{t} \left\| \frac{d^n}{dt^n} u_2(\tau) \right\| d\tau. \end{aligned} \tag{3.50}$$

Here $r_n(t) = \alpha + n(t-a)/(n+1)$. Let $s_n(t) = t + (\beta - t)/(n+1)$. One verifies similarly that

$$\begin{aligned} \left\| \frac{d^n}{dt^n} u_2(t) \right\| &\leqq G_0 G^n M_n (\beta - t)^{-n+1} \\ &+ \frac{CC_1 G_0(\beta-\alpha)^2}{2} L^{n-1}(t-\alpha)^{-n+1}(\beta-t)^{-n+1} M_{n-1} \\ &+ N_0 M_0^2 \int_t^{s_n(t)} \left\| \frac{d^n}{d\tau^n} u_1(\tau) \right\| d\tau. \end{aligned} \tag{3.51}$$

For $i = 1,2$ and $n = 1,2 \cdots$ set $v_i^n(t) = (t-\alpha)^{n-1}(\beta-t)^{n-1} \| (d^n/dt^n) u_i(t) \|$ and observe that by Lemma 3.2 (i), $v_i^n(t)$ is bounded in $[\alpha, \beta]$. Since $(t-\alpha)^{n-1}(\beta-t)^{n-1} \leqq e(\tau-\alpha)^{n-1}(\beta-\tau)^{n-1}$ for $r_n(t) \leqq \tau \leqq s_n(t)$ we find that

$$v_1^n(t) \leqq G_0 G^n M_n (\beta-t)^{n-1} + \frac{CC_1 G_0 (\beta-\alpha)^2}{2} L^{n-1} M_{n-1} + N_0 M_0^2 e \int_\alpha^t v_2^n(\tau) d\tau \tag{3.52}$$

and that

$$v_2^n(t) \leqq G_0 G^n M_n (t-\alpha)^{n-1} + \frac{CC_1 G_0 (\beta-\alpha)^2}{2} L^{n-1} M_{n-1} + N_0 M_0^2 e \int_t^\beta v_1^n(\tau) d\tau. \tag{3.53}$$

Let $\eta_i^n = \sup_{t \in (\alpha,\beta)} v_i^n(t)$. (3.52) and (3.53) guarantee that

$$\eta_1^n \leqq G_0(\beta-\alpha)^{-1}(G(\beta-\alpha))^n M_n + \frac{CC_1}{2L}(\beta-\alpha)^2 G_0 L^n M_n + \tfrac{1}{2}\eta_2^n \tag{3.54}$$

(3.55) $$\eta_2^n \leqq G_0(\beta-\alpha)^{-1}(G(\beta-\alpha))^n M_n + \frac{CC_1}{2L}(\beta-\alpha)^2 G_0 L^n M_n + \tfrac{1}{2}\eta_1^n.$$

Consequently for $i = 1,2$ and $n = 1,2,\cdots$ we have

(3.56) $$\eta_i^n \leqq 2G_0(\beta-\alpha)^{-1}(G(\beta-\alpha))^n M_n + \frac{CC_1}{L}(\beta-\alpha)^2 G_0 L^n M_n.$$

The assumptions on C_1 and L and an induction on n ensure that (i) of Lemma 3.5 is true.

The proof of part (ii) is similar and uses Lemma 3.2 (ii).

Suppose that $A(t)$ satisfies Condition I. In Theorem 3.6 below we assume that $A(t)$ satisfies the following two conditions.

Condition II′. For $i = 1,2$, $E_i(t) \in C^\infty([a,b], B(X,X))$; there exist constants H_0 and H such that for $t \in [a,b]$ and $n = 0,1,\cdots$, we have $\|(d^n/dt^n)E_i(t)\| \leqq H_0 H^n M_n$.

Condition III′. For $i = 1,2$, there exists a complex μ_i such that for $i = 1,2$, the operator $L(t) = (-1)^{i+1}(A(t)E_i(t) + \mu_i I)$ satisfies conditions (i) and (ii) and the following condition:

(iii)′ There exist constants H_0 and H such that (iii) of Condition III is satisfied with $B_n = H_0 H^n M_n$.

It is proved in [9] that if $L(t)$ satisfies (i) and (ii) of Condition III and (iii)′ of Condition III′ and $U(t,\tau)$ is the evolution operator associated with $L(t)$, then there exist constants N_0 and N such that for every pair m,n of non-negative integers and for $a \leqq \tau < t \leqq b$ we have

(3.57) $$\left\| \frac{\partial^n}{\partial t^n}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\right)^m U(t,\tau)\right\| \leqq N_0 N^{m+n} M_{m+n} |t-\tau|^{-n}.$$

Observe that if $L(t)$ satisfies (i), (ii), and (iii)′ and if $B(t) \in C^\infty([a,b], B(X,X))$ and there exist constants H_0 and H such that for $n = 0,1,\cdots$, we have

$$\|(d^n/dt^n)B(t)\| \leqq H_0 H^n M_n$$

then there exists a μ such that $L(t) + B(t) + \mu I$ satisfies (i), (ii), and (iii)′.

THEOREM 3.6. *Suppose that $A(t)$ satisfies Conditions I, II′, and III′. Let $f(t) \in C^\infty[a,b]$ and assume that there exist constants F_0 and F such that for $n = 0,1,\cdots$ and $t \in [a,b]$ we have $\| d^n/dt^n)f(t)\| \leqq F_0 F^n M_n$. Let $u(t)$ be a solution of (1.1) in $[a,b]$. Then $u(t) \in C^\infty(a,b)$ and there exist constants C and G such that for $t \in (a,b)$ and $n = 1,2,\cdots$, we have*

$$\left\| \frac{d^n}{dt^n} u(t) \right\| \leqq CG^n M_n((t-a)^{-n} \| E_1(a)u(a) \| + (b-t)^{-n} \| E_2(b)u(b) \| \tag{3.58}$$
$$+ (t-a)^{-n+1}(b-t)^{-n+1}(|u(t)|_0 + F_0)).$$

PROOF. Since Conditions II′ and III′ hold, the above-mentioned results of [9] gurantee the existence of constants H_0, H, K_0, and K such that estimate (3.32) holds with $B_{m,n} = H_0 H^{m+n} M_m M_n$ and estimate (3.33) holds with

$$C_{m,n} = K_0 K^{m+n} M_m M_n.$$

Theorem 3.3 ensures that $u(t) \in C^\infty(a,b)$. For $i = 1,2$, let $u_i(t) = E_i(t)u(t)$. Let $[\alpha,\beta] \subseteq [a,b]$ and suppose that $K_0 M_0^2(\beta-\alpha)e < \frac{1}{2}$. Let $v_1^j(t)$, $v_2^j(t)$, $j = 1,2,3$, be defined as in the proof of Theorem 3.3. Observe that by Lemma 3.4 (ii) the function $g_1(t) = \int_a^t K_1(t,\tau)E_1(\tau)f(\tau)d\tau$ and $g_2(t) = \int_t^\beta K_2(t,\tau)E_2(\tau)f(\tau)d\tau$ satisfy the requirements of Lemma 3.5 (i) with $G_0 = CF_0$ and some positive constant C. For $t \in [\alpha,\beta]$ we have $u(t) = \sum_{j=1}^3 \sum_{i=1}^2 v_i^j(t)$ and Lemma 3.5(i) and (ii) guarantee the existence of constants C and L such that for $n = 1,2,\cdots$ and $t \in (\alpha,\beta)$ we have

$$\left\| \frac{d^n}{dt^n} u(t) \right\| \leqq CL^n M_n((t-\alpha)^{-n}(\beta-t)^{-n+1} \| E_1(\alpha)u(\alpha) \| \tag{3.59}$$
$$+ (t-\alpha)^{-n+1}(\beta-t)^{-n} \| E_2(\beta)u(\beta) \| + (t-\alpha)^{-n+1}(\beta-t)^{-n+1}F_0).$$

Since (3.59) holds for every $[\alpha,\beta] \subseteq [a,b]$ such that $K_0 M_0^2(\alpha-\beta)e < \frac{1}{2}$ there exist constants C and G such that (3.58) holds for $n = 1,\cdots$ and $t \in (a,b)$.

4. Two point problems for parabolic equations

We consider in this section the equation (1.1) where for each $t \in [a,b]$, $A(t) = A_B^p(t)$ is the unbounded operator in $H^{0,p}(G)$ that is associated with an elliptic $l \times l$ differential system $A(t,x,D)$ of order ω independent of t and with the boundary operators $B_j(t,x,D)$ for $j = 1,\cdots,\frac{1}{2}\omega l$. Thus for each $t \in [a,b]$, $A(t,x,D)$ and $B_j(t,x,D)$ respectively satisfy the assumption on $A(x,D)$ and on $B_j(x,D)$ of Section 1. For $t \in [a,b]$, $D(A_B^p(t)) = H^{\omega,p}(G\{B_j(t)\})$ and for $u \in D(A_B^p(t))$ we have $A_B^p(t)u = A(t,x,D)u$.

For the duration of this work we use the notation and definitions of [3]. It is proved in [3] that if $A(t,x,D)$ and $B_j(t,x,D)$ for $j = 1,\cdots,\frac{1}{2}\omega l$ satisfy Agmon's conditions on the rays $l\pi/2$ and $(l-\pi)/2$, then there exist bounded projections $E_1(t)$ and $E_2(t)$ in $H^{0,p}(G)$ such that $A_B^p(t)E_1(t)$ and $-A_B^p(t)E_2(t)$ are infinitesimal generators of analytic semigroups and $A_B^p(t)$ is completely reduced by the direct sum decomposition

$$H^{0,p}(G) = \sum_{i=1}^{2} \oplus E_i(t)H^{0,p}(G).$$

Observe that Agmon's conditions for $A(x,D)$, $B_j(x,D)$ for $j = 1, \cdots, \frac{1}{2}\omega l$, and the ray l_θ guarantee the existence of constants C and R such that for $\lambda \in l_\theta$ with $|\lambda| \geqq R$ we have $\lambda \in \rho(A_B^p)$ and $\|(\lambda - A_B^p)^{-1}\| \leqq C/|\lambda|$. (See [1] and [7].) In this section we show that with adequate regularity assumptions on the coefficients of $A(t,x,D)$ and $B_j(t,x,D)$ for $j = 1, \cdots, \frac{1}{2}\omega l$, locally $E_1(t)$ and $E_2(t)$ may be chosen so that $A_B^p(t)$ satisfies the assumptions of the theorems in Section 2.

As in [3], denote by $P_0(\lambda, t)$ the zeroth order parametrix for $(\lambda - A_B^p(t))^{-1}$ of Seeley [7] that is well defined for $\lambda \in l_\theta$ provided that $A(t,x,D)$ and $B_j(t,x,D)$ satisfy Agmon's conditions on l_θ. Recall that

$$P_0(\lambda, t) = \sum_{j=1}^{N} C_j(\lambda, t) - \sum_{j=m+1}^{N} D_j(\lambda, t) \tag{4.1}$$

and for $f \in C_0^\infty(G)$, $C_j(\lambda, t)f$ and $D_j(\lambda, t)f$ are given in terms of local coordinates by

$$C_j(\lambda, t)f(x) = (2\pi)^{-\nu}\psi_j(x) \int e^{ix\xi}\theta(\xi, \lambda)(\sigma_\omega A(t,x,\xi) - \lambda)^{-1}\widetilde{\phi_j f}(\xi)d\xi \tag{4.2}$$

and

$$\begin{aligned} &D_j(\lambda, t)f(x) \\ &= (2\pi)^{-\nu+1}\psi_j(x) \int e^{ix'\xi'}\theta'(\xi', \lambda)d(t, x', x_\nu, \xi', s, \lambda)\widetilde{\phi_j f}_\nu(\xi', s)d\xi' ds. \end{aligned} \tag{4.3}$$

For $g \in C_0^\infty(R^\nu)$ we use the notation $\tilde{g}(\xi) = \int_{R^\nu} e^{ix\xi}g(x)dx$ and

$$\tilde{g}_\nu(\xi', x_\nu) = \int_{R^{\nu-1}} e^{ix'\xi'}g(x)dx'.$$

The scalar functions $\phi_j(x)$ and $\psi_j(x)$, $j = 1, \cdots, N$, are in $C^\infty(\bar{G})$ and for $j = 1, \cdots, m$, the support of $\phi_j(x)$ and $\psi_j(x)$ is disjoint from $\partial\bar{G}$. $\theta(\xi, \lambda)$ is infinitely differentiable in $R^\nu \times l_\theta$ where $\theta(\xi, \lambda) = 0$ for $|\xi|^2 + |\lambda|^2 \leqq \frac{1}{2}$ and $\theta(\xi, \lambda) = 1$ for $|\xi|^2 + |\lambda|^2 \geqq 1$. Similarly $\theta'(\xi', \lambda)$ is infinitely differentiable in $R^{\nu-1} \times l_\theta$ where $\theta'(\xi', \lambda) = 0$ for $|\xi'|^2 + |\lambda|^2 \leqq \frac{1}{2}$ and $\theta'(\xi', \lambda) = 1$ for $|\xi'| + |\lambda|^2 \geqq 1$. $\sigma_\omega A(t,x,\xi)$ is the symbol of the principal part of $A(t,x,D)$. Recall that if Agmon's conditions are satisfied by $A(x,D)$ and $B_j(x,D)$, $j = 1, \cdots, \frac{1}{2}\omega l$ on l_θ, then $\sigma_\omega A(x,\xi) - \lambda$ is a regular matrix for $(\xi, \lambda) \in R^\nu \times l_\theta$ such that $|\xi|^2 + |\lambda|^2 \neq 0$. Also for every set of integers l, m, and p and for every pair of multi-indices δ' and ε' of length $\nu - 1$ there exist constants c and c_1 such that for $x_\nu \geqq 0$, $s \geqq 0$, and for $(\xi', \lambda) \in R^{\nu-1} \times l_\theta$ such that $|\xi'|^2 + |\lambda|^2 \neq 0$ we have

$$
(4.4) \qquad \begin{aligned} &\left| D_{x'}^{\delta'} D_{\xi'}^{\gamma'} s^l \frac{\partial^m}{\partial x_\nu^m} \frac{\partial^p}{\partial \lambda^p} \tilde{d}(x', x_\nu, \xi', s, \lambda) \right| \\ &\leqq c \exp(-c_1(x_\nu + s)(|\xi'| + |\lambda|^{1/\omega}))(|\xi'| + |\lambda|^{1/\omega})^{1-\omega-|\varepsilon'|-\omega p+m-l}. \end{aligned}
$$

(See [7].)

LEMMA 4.1. *Suppose that any of the coefficients $a(t,x)$ of $A(t,x,D)$ is infinitely differentiable in $[a,b] \times \bar{G}$ and that for every multi-index α there exist constants H_0 and H such that for $n = 0, 1 \cdots$ and $(t,x) \in [a,b] \times \bar{G}$ we have*

$$
\left| \frac{\partial^n}{\partial t^n} \frac{\partial^\alpha}{\partial x^\alpha} a(t,x) \right| \leqq H_0 H^n M_n.
$$

(i) *Let $\pi/2 < \alpha_1 < \alpha_2 < 3\pi/2$. Suppose that $\sigma_\omega A(t,x,\xi) - \lambda$ is regular for $(t,x) \in [a,b] \times \bar{G}$ and $(\xi,\lambda) \in R^\nu \times \gamma(\alpha_1,\alpha_2)$ such that $|\xi|^2 + |\lambda|^2 \neq 0$. Let $1 \leqq j \leqq N$. Then*

$$
\lim_{\tau \to 0} \frac{1}{2\pi i} \int_{\gamma(\alpha_1,\alpha_2)} e^{\lambda\tau} C_j(\lambda, t) d\lambda f
$$

exists for every $f \in H^{0,p}(G)$. For $t \in [a,b]$ and $f \in H^{0,p}(G)$ set

$$
Q_j(t) f = \lim_{\tau \to 0} \int_{\gamma(\alpha_1,\alpha_2)} e^{\lambda\tau} C_j(\lambda, t) d\lambda f.
$$

Then there exist constants H_0 and H such that

$$
Q_j(t) \in G(H_0, H, [a,b], B(H^{0,p}(G), H^{0,p}(G))).
$$

(ii) *Let $0 < \alpha_1 < \alpha_2 < 2\pi$ and assume that $\sigma_\omega A(t,x,\xi) - \lambda$ is regular for $(t,x) \in [a,b] \times \bar{G}$ and $(\xi,\lambda) \in R^\nu \times l_\theta$ such that $|\xi|^2 + |\lambda|^2 \neq 0$. Let $1 \leqq j \leqq N$. For $\mu \notin \gamma(\alpha_1,\alpha_2)$ set*

$$
S_j(\mu, t) = \frac{1}{2\pi i} \int_{\gamma(\alpha_1,\alpha_2)} C_j(\lambda, t)(\mu - \lambda)^{-1} d\lambda.
$$

Then for $t \in [a,b]$, $S_j(\mu,t)$ is analytic in the complement of $\gamma(\alpha_1,\alpha_2)$. For every sufficiently small $\varepsilon > 0$ there exist constants H_0 and H such that for

$$
\mu \notin \Gamma(\alpha_1 - \varepsilon, \alpha_1 + \varepsilon) \cap \Gamma(\alpha_2 - \varepsilon, \alpha_2 + \varepsilon)
$$

we have

$$S_j(\mu, t) \in G(|\mu|^{-1}H_0, H, [a,b], B(H^{0,p}(G), H^{0,p}(G))).$$

(iii) *Suppose that* $\sigma_\omega A(t,x,D) - \lambda$ *is regular for* $(t,x) \in [a,b] \times \bar{G}$ *and* $(\xi, \lambda) \in R^\nu \times l_\theta$ *such that* $|\xi|^2 + |\lambda|^2 \neq 0$. *Let* $1 \leqq j \leqq N$. *There exist constants* H_0 *and* H *such that for* $\lambda \in l_\theta$ *we have* $C_j(\lambda, t) \in G(H_0, H, [a,b], B(H^{0,p}(G), H^{\omega,p}(G)))$ *and* $|\lambda| C_j(\lambda, t) \in G(H_0, H, [a,b], B(H_0^p(G), H_0^p(G)))$.

PROOF. Let $1 < p < \infty$ and let $0 < R < \infty$. Suppose that $K(x,\xi)$ vanishes for $|x| \geqq R$ and that $|\xi|^{|\beta|} |D_x^\alpha D_\xi^\beta k(x,\xi)| \leqq 1$ for $|\alpha| \leqq \nu - 1$ and $|\beta| \leqq \nu$. [8, Lem. 1] ensures the existence of a constant $C = C(p, v, R)$ such that for every $f \in C_0^\infty(G)$ we have

$$\| (2\pi)^{-\nu} \int e^{ix\xi} k(x\xi) \tilde{f}(\xi) d\xi \|_{L_p(R^\nu)} \leqq C \| f \|_{L_p(R^\nu)}. \tag{4.5}$$

To prove part a set

$$a_1(t,x,\xi,\tau) = \frac{\psi(x)}{2\pi i} x(\xi) \int_{L^-(|\xi|)} e^{\lambda\tau} (\sigma_\omega A(t,x,\xi) - \lambda)^{-1} d\lambda.$$

Here $\tau \geqq 0$ and for $|\xi| > 0$, $L^-(|\xi|)$ is the boundary of

$$\Gamma(\alpha_1, \alpha_2) \cap \{\lambda; r_1|\xi| \leqq |\lambda|^{1/\omega} \leqq r_2|\xi|\};$$

r_i, $i = 1,2$, are chosen so that for all x and $\xi \neq 0$ and for $t \in [a,b]$ the eigenvalues of $\sigma_\omega A(t,x,\xi)$ are contained in $\Gamma(\alpha_1,\alpha_2) \cap \{\lambda; 2r_1|\xi| \leqq |\lambda|^{1/\omega} \leqq \frac{1}{2} r_2 |\xi|\}$. $\chi(\xi)$ is infinitely differentiable in R^ν; $\chi(\xi) = 0$ for $|\xi| \leqq 1$ and $\chi(\xi) = 1$ for $|\xi| \geqq 2$. Let $a_2(t,x,\xi,\tau) = 0$ for $|\xi| \geqq 2$ and for $|\xi| \leqq 2$ set

$$a_2(t,x,\xi,\tau) = \frac{1}{2\pi i} \psi(x)(1 - \chi(\xi)) \int e^{\lambda\tau} \theta(\xi,\lambda)(\sigma_\omega A(t,x,\xi) - \lambda)^{-1} d\lambda$$

where ∂ is the boundary of $\Gamma(\alpha_1,\alpha_2) \cap \{\lambda; |\lambda|^{1/\omega} \leqq 2r_2\}$. Finally for $\xi \neq 0$ and $\tau \geqq 0$ set

$$a_3(t,x,\xi,\tau) = \frac{1}{2\pi i} \psi(x)\chi(\xi) \int_{L^-(|\xi|)} e^{\lambda\tau} \lambda^{-1} \sigma_\omega (A(t,x,\xi) - \lambda)^{-1} d\lambda.$$

Now

$$|\xi|^\beta D_x^\alpha D_\xi^\beta (a_3(t,x,\xi,\tau) - a_3(t,x,\xi,0))$$

$$= \frac{1}{2\pi i} |\xi|^\beta D_x^\alpha \psi(x) \chi(\xi) \int_{L^-(|\xi|)} \frac{e^{\lambda\tau} - 1}{\lambda} D_\xi^\beta (\sigma_\omega A(t,x,\xi) - \lambda)^{-1} d\lambda \tag{4.6}$$

$$+ \frac{1}{2\pi i} |\xi|^\beta \sum_{\substack{|\varepsilon| \neq 0 \\ \varepsilon \leqq \beta}} \binom{\beta}{\varepsilon} D_x^\alpha \psi(x) D_\xi^\varepsilon \chi(\xi) \int_{L^-(|\xi|)} \frac{e^{\lambda\tau} - 1}{\lambda} D_\xi^{\beta - \varepsilon} (\sigma_\omega A(t,x,\xi) - \lambda)^{-1} d\lambda.$$

Considerations of homogeneity imply that the first term on the right-hand side of (4.6) tends to zero as τ tends to zero uniformly with respect to $(t,x)\in[a,b]\times\bar{G}$ and $\xi\in R^\nu$. Since $D^\alpha\chi(\xi)=0$ for $|\alpha|\neq 0$ and ξ such that $|\xi|\leqq 1$ or $|\xi|\geqq 2$, the same result also holds for the second term on the right-hand side of (4.6). Consequently for $i=3$ we have

$$\lim_{\tau\to 0}\left|\xi\right|^p\left|D_x^\alpha D_\xi^\beta(a_i(t,x,\xi,\tau)-a_i(t,x,\xi,0))\right|=0$$

uniformly with respect to $(t,x)\in[a,b]\times\bar{G}$ and $\xi\in R^\nu$. The same result holds also for $i=2$. Note that for $f\in C_0^\infty(G)$ and $\tau>0$ we have

$$\text{(4.7)}\qquad \frac{1}{2\pi i}\int_{\gamma(\alpha_1,\alpha_2)}e^{\lambda\tau}C_j(\lambda,t)d\lambda f(x)=\sum_{i=1}^{2}\int e^{ix\xi}a_i(t,x,\xi,\tau)\widetilde{\phi_j f}(\xi)d\xi$$

and for $\tau\geqq 0$ we have

$$\text{(4.8)}\qquad \int e^{ix\xi}a_1(t,x,\xi,\tau)\widetilde{\phi_j f}(\xi)d\xi=\int e^{ix\xi}a_3(t,x,\xi,\tau)\widetilde{A(t,x,D)\phi_j f}(\xi)\xi.$$

Let $f\in C_0^\infty(G)$, (4.7), (4.8), and [8, Lem. 1] that we have cited above guarantee the existence of

$$\lim_{\tau\to 0}\frac{1}{2\pi i}\int_{\gamma(\alpha_1,\alpha_2)}e^{\lambda\tau}C_j(\lambda,\tau)d\lambda f$$

and that

$$\text{(4.9)}\qquad \lim_{\tau\to 0}\frac{1}{2\pi i}\int_{\gamma(\alpha_1,\alpha_2)}e^{\lambda\tau}C_j(\lambda,\tau)\,d\lambda f(x)=\sum_{i=1}^{2}\int e^{ix\xi}a_i(t,x,\xi,0)\widetilde{\phi_j f}(\xi)d\xi.$$

To ensure the existence of $\lim_{\tau\to 0}\int e^{\lambda\tau}C_j(\lambda t)\,d\lambda f$ for every $f\in H^{0,p}(G)$, it is now sufficient to observe that by [3, Lem. 4.2], for every $t\in[a,b]$ there exists a constant C_t such that

$$\left\|\frac{1}{2\pi i}\int_{\gamma(\alpha_1,\alpha_2)}e^{\lambda\tau}C_j(\lambda,t)\,d\lambda\right\|\leqq C_t$$

for $\tau>0$. Note that it follows from the assumptions on $\{M_n\}$ and on $\sigma_\omega A(t,x,\xi)$ that for every pair of multi-indices α and β there exist constants H_0 and H such that for $(t,x)\in[a,b]\times\bar{G}$, $\xi\neq 0$ and $\lambda\in L^-(|\xi|)$ we have

$$\text{(4.10)}\qquad \left|D_x^\alpha D_\xi^\beta\frac{\partial^n}{\partial t^n}(\sigma_\omega A(t,x,\xi)-\lambda)^{-1}\right|\leqq H_0H^mM_n(|\xi|+|\lambda|^{1/\omega})^{-\omega-|\beta|}.$$

(4.10), (4.9), the definition of $a_i(t,x,\xi,0)$, and the above-mentioned [8, Lem. 1] guarantee the existence of constants H_0 and H such that

$$Q_j(t) \in G(H_0, H, [a,b], B(H^{0,p}(G), H^{0,p}(G))).$$

Let $b_1(t,x,\xi,\mu) = a_1(t,x,\xi,0)(\sigma_\omega A(t,x,\xi) - \mu)^{-1}$. Set $b_2(t,x,\xi,\mu) = 0$ for $|\xi| \geqq 2$ and for $|\xi| < 2$ set

$$b_2(t,x,\xi,\mu) = \frac{1}{2\pi i}\psi(x)(1-\chi(\xi))\int_\partial \theta(\xi,\lambda)(\sigma_\omega A(t,x,\xi) - \lambda)^{-1}(\mu - \lambda)^{-1}d\lambda.$$

Then for $f \in C_0^\infty(G)$ we have

$$S_j(\mu,t)(x) = (2\pi)^{-\nu}\sum_{i=1}^{2}\int e^{ix\xi}b_i(t,x,\xi,\mu)\widetilde{\phi_j f}(\xi)d\xi$$

and arguments similar to those used above ensure the validity of (ii); (iii) is checked similarly.

Lemma 4.2. *Suppose that any one of the coefficients $a(t,x)$ of $A(t,x,D)$ or of $B_j(t,x,D)$, $j = 1,\cdots,\frac{1}{2}\omega l$, is infinitely differentiable in $[a,b]\times\bar{G}$. Assume that for every multi-index α there exist constants H_0 and H such that for $n = 0,1\cdots$ and $(t,x)\in[a,b]\times\bar{G}$ we have $|(\partial^n/\partial t^n)a(t,x)| \leqq H_0H^nM_n$. Assume that for $t\in[a,b]$, $A(t,x,D)$ and $B_j(t,x,D)$, for $j=1,\cdots,\frac{1}{2}\omega l$, satisfy Agmon's conditions on the ray l_θ. Let $m+1\leqq j\leqq N$. Then there exists a constant c_1 such that for every pair of multi-indices δ' and ε' and for every set of integers l, m, and p there exist constants H_0 and H such that for $t\in[a,b]$ and $n = 0,1\cdots$ we have*

$$(4.11)\qquad D_{x'}^{\delta'}D_{\xi'}^{\varepsilon'}s^l\frac{\partial^m}{\partial x^p}\frac{\partial^p}{\partial\lambda!}\frac{\partial^n}{\partial t^n}\widetilde{d_j}(t,x',x_\nu,\xi',s,l)$$

$$\leqq H_0H^nM_n\exp(-c_1(x_\nu+s)(|\xi'|+|\lambda|^{1/\omega}))(|\xi'|+|\lambda|^{1/\omega})^{1-\omega-|\varepsilon'|-l+m-\omega p}.$$

Note that for $A(t,x,D)$ and $B_j(t,x,D)$ independent of t and for $n=0$, estimate (4.11) coincides with the above-mentioned estimate (4.4). We can use the same proof as in [7, (3.4)] with obvious modifications to derive estimate (4.11). [7, Lem. 3, (II)] is used to investigate the dependence of $d(t,x',x_\nu,\xi',s,\lambda)$ on t.

Lemma 4.3. *Let the assumptions of Lemma 4.2 be satisfied.*

(i) *There exist constants H_0 and H such that for $\lambda\in l_\theta$*

$$D_j(\lambda,t)\in G(H_0,H,[a,b],B(H^{0,p}(G),H^{\omega,p}(G)))\text{ and}$$

$$|\lambda|D_j(\lambda,t)\in G(H_0,H,[a,b],B(H^{0,p}(G),H^{0,p}(G))).$$

(ii) $\lim_{\tau\to 0}\int_{l_\theta}e^{\lambda\tau}D_j(\lambda,t)d\lambda f$ *exists for every $f\in H^{0,p}(G)$ and $t\in[a,b]$. For $f\in H^{0,p}(G)$ set $P_j(t=\lim_{\tau\to 0}\int_{l_\theta}e^{\lambda\tau}D_j(\lambda,t)d\lambda f$. There exist constants H_0 and H such that $P_j(t)\in G(H_0,H,[a,b],B(H^{0,p}(G),H^{0,p}(G)))$.*

(iii) *For* $\mu \notin l_\theta$, *set*

$$W_j(\mu, t) = \frac{1}{2\pi i} \int_{l_\theta} D_j(\lambda, t)(\mu - \lambda)^{-1} d\lambda.$$

For $t \in [a, b]$, $W_j(\mu, t)$ *is analytic in the complement of* l_θ. *For every sufficiently small* $\varepsilon > 0$ *there exist constants* H_0 *and* H *such that for* $\mu \notin \Gamma(\theta - \varepsilon, \theta + \varepsilon)$ *we have*

$$W_j(\mu, t) \in G(|\mu|^{-1} H_0, H, [a, b], B(H^{0,p}(G), H^{0,p}(G))).$$

PROOF. Let $1 < p < \infty$. Suppose that $k(x', x_\nu, \xi', s)$ has support in $|x'| \leqq R$ and satisfies the estimate $|\xi'|^{|\beta'|} |D_{x'}^{\alpha'} D_{\xi'}^{\beta'} k(x', x_\nu, \xi', s)| \leqq (x_\nu + s)^{-1}$ for $s > 0$, $x_\nu > 0$, $|\beta'| \leqq \nu$, and $|\alpha'| \leqq \nu$. [8, Lem. 2] ensures the existence of a constant $C = C(p, \nu, R)$ such that the estimate

$$\left\| (2\pi)^{-\nu+1} \int e^{ix'\xi'} k(x', x_\nu, \xi', s) \tilde{g}_\nu(\xi', s) d\xi', ds \right\|_{L_p(R^\nu)} \leqq \| g \|_{L_p(R^\nu)}$$

holds for every $g \in C_0^\infty(R^\nu)$ with support in the interior of R_+^ν. This result and estimate (4.11) with $s = 0$, $l = 0$, $p = 0$, $|\varepsilon'| \leqq \delta$, $|\delta'| \leqq \nu + i$, $m \leqq k$, and $i + k \leqq \omega$ imply that (i) is true.

Let $f \in C_0^\infty(G)$. Then using local coordinates as in (4.3) we find that

$$\begin{aligned} &D_j(\lambda, t) f(x) \\ &= (2\pi)^{-\nu+1} \psi_j(x) \int e^{ix'\xi'} \theta'(\xi', \lambda) s \tilde{d}(t, x', x_\nu, \xi', s, \lambda) \phi_j s^{-1} f_\nu(\xi', s) d\xi' ds. \end{aligned} \tag{4.12}$$

Estimate (4.11) with $l = 1$, $|\delta'| \leqq \nu$, $|\varepsilon'| \leqq \nu$, $m = p = 0$ and [8, Lem. 2] (that we have cited above) ensure that for every $f \in C_0^\infty(G)$ and for $n = 0, 1 \cdots$ there exists a constant $C_{(n,f)}$ such that for $t \in [a, b]$ we have

$$\left\| \frac{\partial^n}{\partial t^n} D_j(\lambda, t) f \right\| \leqq C(n, f)(1 + |\lambda|^{1 + 1/\omega})^{-1}.$$

Consequently

$$\lim_{\tau \to 0} \frac{1}{2\pi i} \int_{l_\theta} e^{\lambda\tau} \frac{\partial^n}{\partial t^n} D_j(\lambda, t) d\lambda f$$

exists for $f \in C_0^\infty(G)$.

Suppose that $k(x', x_\nu, \xi', s, r)$ vanishes for $|x'| \leqq R$ and that there exists a constant c such that $|D_{x'}^{\alpha'} D_{\xi'}^{\beta'} k(x', x_\nu, \xi', s, r)| \leqq$

$\exp(-c(x_\nu + s)(|\xi'| + r))|\xi'| + r) \cdot (|\xi'| + r)^{1-\omega-|\beta'|}$ for $|\alpha'| \leqq \nu$ and $|\beta'| \leqq \nu$

and with $\omega \geqq 1$. [8, Lem. 3] ensures the existence of a constant $C = C(p, c, R, \nu)$ such that for $g \in C_0^\infty(R^\nu)$ with support in interior of R_+^ν we have

$$\left\|(2\pi)^{-\nu+1}\int e^{ix'\xi'}\left(\int_{r_1}^{r_2} r^{\omega-1}k(x', x_\nu, \xi', s, r)dr\right)\tilde{g}_\nu(x', s)d\xi' ds\right\|_{L_p(R_+^\nu)}$$
$$\leqq C\|g\|_{L_p(R_+^\nu)}.$$

For $\tau > 0$ and $n = 0, 1, \cdots$ set

$$P_{n,j}(t,\tau) = \frac{1}{2\pi i}\int_{l_0} e^{\lambda\tau}\frac{\partial^n}{\partial t^n}D_j(\lambda, t)d\lambda.$$

Estimate (4.11) with $|\varepsilon'| \leqq \nu$, $|\delta'| \leqq \nu$, $m = l = p = 0$, and [8, Lem. 3] (that we have cited above) ensure the existence of constants H_0 and H such that for $t \in [a, b]$, $\tau > 0$, and $n = 0, 1, \cdots$, we have

$$\|P_{n,j}(t,\tau)\| \leqq H_0 H^n M_n. \tag{4.13}$$

Since, as we have checked above, $\lim_{\tau\to 0} P_{n,j}(t,\tau)f$ exists for every $f \in C_0^\infty(G)$ we conclude, using (4.13), that $\lim_{\tau\to 0} P_{n,j}(t,\tau)f$ exists for every $f \in H^{0,p}(G)$. It also follows from (4.13) that there exist constants H_0 and H such that

$$P_j(t) \in G(H_0, H, [a, b], B(H^{0,p}(G), H^{0,p}(G)).$$

Part (iii) is proved similarly.

An immediate consequence of Lemmas 4.1 and 4.3 and of the relation (4.1) is the following lemma.

LEMMA 4.4. *Suppose that the assumptions of Lemma 4.2 are satisfied for* $\theta = \alpha_i$, $i = 1, 2$, *with* $\pi/2 < \alpha_1 < \alpha_2 < 3\pi/2$.

(i) *There exist constants* H_0 *and* H *such that for* $i = 1, 2$ *and* $\lambda \in l_{\alpha_i}$ *we have*

$$P_0(\lambda, t) \in G(H_0, H, [a, b], B(H^{0,p}(G), H^{\omega,p}(G)))$$

and

$$|\lambda| P_0(\lambda, t) \in G(H_0, H, [a, b], B(H^{0,p}(G), H^{0,p}(G))).$$

(ii) $\lim_{\tau\to 0} 1/(2\pi i)\int_{\gamma(\alpha_1,\alpha_2)} e^{\lambda\tau}P_0(\lambda, t)d\lambda f$ *exists for every* $f \in H^{0,p}(G)$. *For* $f \in H^{0,p}(G)$ *set*

$$B(t)f = \lim_{\tau\to 0}\frac{1}{2\pi i}\int_{\gamma(\alpha_1,\alpha_2)} e^{\lambda\tau}P_0(\lambda, t)d\lambda f.$$

There exist constants H_0 *and* H *such that*

$$B(t) \in G(H_0, H, [a, b], B(H^{0,p}(G), H^{0,p}(G))).$$

(iii) *For* $\mu \notin \gamma(\alpha_1, \alpha_2)$ *let*

$$R(\mu, t) = \frac{1}{2\pi i}\int_{\gamma(\alpha_1,\alpha_2)} P_0(\lambda, t)(\mu-\lambda)^{-1}d\lambda.$$

For $t \in [a, b]$, $R(\mu, t)$ *is analytic in the complement of* $\gamma(\alpha_1, \alpha_2)$. *For every sufficiently small* $\varepsilon > 0$ *there exist constants* H_0 *and* H *such that for*

$$\mu \notin \Gamma(\alpha_1 - \varepsilon, \alpha_1 + \varepsilon) \cup \gamma(\alpha_2 - \varepsilon, \alpha_2 + \varepsilon)$$

we have $R(\mu, t) \in G(|\mu|^{-1}H_0, H, [a, b], B(H^{0,p}(G), H^{0,p}(G)))$.

LEMMA 4.5. *Let the assumptions of Lemma* 4.2 *be satisfied.*

(i) *For* $\lambda \in l_\theta$ *and* $t \in [a, b]$ *let* $W(\lambda, t)$ *be the bounded operator in* $H^{0,p}(G)$ *such that for* $f \in C_0^\infty(G)$, $W(\lambda, t)f = ((\lambda - A(t))P_0(\lambda, t) - I)f$. *There exist constants* H_0 *and* H *such that for* $\lambda \in l_\theta$ *we have*

$$(1 + |\lambda|^{1-\omega})W(\lambda, t) \in G(H_0, H, [a, b], B(H^{0,p}(G), H^{0,p}(G))).$$

(ii) *For* $\lambda \in l_\theta$, $t \in [a, b]$, *and* $j = 1, \cdots, \frac{1}{2}\omega l$ *let* $G_j(\lambda, t)$ *be the bounded operator from* $H^{0,p}(G)$ *to* $H^{\omega-\omega_j,p}(G)$ *such that for* $f \in C_0^\infty(G)$, $G_j(\lambda, t)f = B_j(t)P_0(\lambda, t)f$. *There exist constants* H_0 *and* H *such that for* $\lambda \in l_\theta$ *we have*

$$G_j(\lambda, t) \in G(H_0, H, [a, b], B(H^{0,p}(G), H^{\omega-\omega_j,p}(G)))$$

and

$$|\lambda|^{1-\omega_j/\omega}G_j(\lambda t) \in G(H_0, H, [a, b], B(H^{0,p}(G), H^{0,p}(G))).$$

The proof of this lemma is similar to the proofs of [7, Lem. 4, 5, and 6].

The following results are proved in Tanabe [9]. Let the assumptions of Lemma 4.2 be satisfied for θ such that $\alpha_1 \leqq \theta \leqq \alpha_2$. There exists a constant R such that $\lambda \in \rho(A_B^p(t))$ for $\lambda \in \Gamma(\alpha_1, \alpha_2)$ with $|\lambda| \geqq R$ and for $t \in [a, b]$,

$$(\lambda - A_B^p(t))^{-1} \in C^\infty([a, b], B(H^{0,p}(G), H^{\omega,p}(G)))$$

for $\lambda \in \Gamma(\alpha_1, \alpha_2)$ with $|\lambda| \geqq R$ and there exist constants H_0 and H such that for $t \in [a, b]$, $\lambda \in \Gamma(\alpha_1, \alpha_2)$ with $|\lambda| \geqq R$ and $n = 0, 1 \cdots$ we have

$$\left\|\frac{\partial^n}{\partial t^n}(\lambda - A_B^p(t))^{-1}\right\|_\omega + |\lambda|\left\|\frac{\partial^n}{\partial t^n}(\lambda - A_B^p(t))\right\|_0 \leqq H_0 H^n M_n. \tag{4.14}$$

These results are proved in [9] for $l = 1$ and the same proof applies also to systems.

LEMMA 4.6. *Let the assumptions of Lemma* 4.2 *be satisfied. There exist constants* H_0, H, *and* R *such that for* $t \in [a, b]$, $\lambda \in l_\theta$ *with* $|\lambda| \geqq R$ *and* $n = 0, 1 \cdots$ *we have* $\lambda \in \rho(A_B^p(t))$ *and*

$$\left\|\frac{\partial^n}{\partial t^n}((\lambda - A_B^p(t))^{-1} - P_0(\lambda,t))\right\|_\omega$$

$$(4.15)\qquad + |\lambda| \left\|\frac{\partial^n}{\partial t^n}(\lambda - A_B^p(t))^{-1} - P_0(\lambda,t)\right\|_0$$

$$\leqq H_0 H^n M_n (1 + |\lambda|^{1/\omega})^{-1}.$$

Proof. Choose R so that $\lambda \in \rho(A_B^p(t))$ provided that $t \in [a,b]$, $\lambda \in l_\theta$, and $|\lambda| \geqq R$. For t, λ as above, set $u(t,\lambda) = ((\lambda - A_B(t))^{-1} - P_0(\lambda,t))f$. Then

$$(4.16)\qquad (\lambda - A(t))u(t,\lambda) = W(t,\lambda)f$$

and for $j = 1,\cdots,\frac{1}{2}\omega l$

$$(4.17)\qquad B_j(t)u(t,\lambda) = G_j(t,\lambda)f \text{ on } \partial G$$

where $W(t,\lambda)$ and $G_j(t,\lambda)$, $j = 1,\cdots,\frac{1}{2}\omega l$, are defined as in Lemma 4.5. Using Lemmas 4.4 and 4.5 and the above-mentioned results of Tanabe we find that

$$(4.18)\quad (\lambda - A(t))\frac{\partial^n}{\partial t^n}u(t,\lambda) = -\sum_{i=0}^{n-1}\binom{n}{i}A^{n-i}(t)\frac{\partial^i}{\partial t^i}u(t,\lambda) + \frac{\partial^n}{\partial t^n}W(t,\lambda)f$$

and on ∂G

$$(4.19)\quad B_j(t)\frac{\partial^n}{\partial t^n}u(t,\lambda) = -\sum_{i=0}^{n-1}\binom{n}{i}B_j^{n-i}(t)\frac{\partial^i}{\partial t^i}u(t,\lambda) + \frac{\partial^n}{\partial t^n}G_j(t,\lambda)f.$$

Here $A^k(t)$ denotes the differential system obtained from $A(t,x,D)$ by differentiating each of the coefficients of $A(t,x,D)$, with respect to t, k times; $B_j^k(t)$ is defined similarly.

Observe that there exist constants C and R such that

$$(4.20)\quad \|v\|_\omega + |\lambda|\,\|v\|_0 \leqq C(\|f\|_0 + \sum_{j=1}^{\frac{1}{2}\omega l}\|g_j\|_{\omega-\omega_j} + |\lambda|^{1-\omega_j/\omega}\|g_j\|_0)$$

provided that $t \in [a,b]$, $v \in H^{\omega,p}(G)$, $(A(t) - \lambda)v = f$, $g_j \in H^{\omega-\omega_j,p}(G)$ for $j = 1,\cdots,\frac{1}{2}\omega l$, $B_j(t)v = g_j$ on ∂G for $j = 1,\cdots,\frac{1}{2}\omega l$, $\lambda \in l_\theta$, and $|\lambda| \geqq R$ (see [9]). The assumptions of the present lemma, Lemma 4.5 and the *a priori* estimate (4.20) guarantee the existence of constants B_0 and B such that for $n = 1,2,\cdots$ and $t \in [a,b]$

$$\left\| \frac{\partial^n}{\partial t^n} u(t,\lambda) \right\|_\omega + |\lambda| \left\| \frac{\partial^n}{\partial t^n} u(t,\lambda) \right\|_0$$

$$(4.21) \leqq \sum_{i=0}^{n-1} \binom{n}{i} B_0 B^{n-i} M_{n-i} \left\| \frac{\partial^i}{\partial t^i} u(t,\lambda) \right\|_\omega + B_0 B^n M_n (1 + |\lambda|^{1-\omega})^{-1} \| f \|_0$$

$$+ \sum_{j=1}^{\frac{1}{2}\omega l} |\lambda|^{1-\omega_j/\omega} \sum_{i=0}^{n-1} \binom{n}{i} B_0 B^{n-i} M_{n-i} \left\| \frac{\partial^i}{\partial t^i} u(t,\lambda) \right\|_{\omega_j}.$$

Estimate (4.21) and the estimate $\| f \|_{\omega_j} \leqq C \| f \|_\omega^{\omega_j/\omega} \| f \|_0^{1-\omega_j/\omega}$ that holds for every $f \in H^{\omega, p}(G)$ and $j = 1, \cdots, \frac{1}{2}\omega l$ with an appropriate constant C, guarantee the existence of constants B_0 and B such that for $n = 1, 2, \cdots$ and $t \in [a, b]$ we have

$$\left\| \frac{\partial^n}{\partial t^n} u(t,\lambda) \right\|_\omega + |\lambda| \left\| \frac{\partial^n}{\partial t^n} u(t,\lambda) \right\|_0$$

$$(4.22) \qquad \leqq \sum_{i=0}^{n-1} \binom{n}{i} B_0 B^{n-i} M_{n-i} \left(\left\| \frac{\partial^i}{\partial t^i} u(t,\lambda) \right\|_\omega + |\lambda| \left\| \frac{\partial^i}{\partial t^i} u(t,\lambda) \right\|_0 \right)$$

$$+ \; B_0 B^n M_n (1 + |\lambda|^{1/\omega})^{-1} \| f \|_0.$$

The relations (4.16) and (4.17), Lemma 4.5, and the *a priori* estimate (4.20) ensure the existence of a constant $H_0 > 2B_0$ such that

$$(4.23) \qquad \| u(t,\lambda) \|_\omega + |\lambda| \, \| u(t,\lambda) \|_0 \leqq H_0 M_0 (1 + |\lambda|^{1/\omega})^{-1} \| f \|_0.$$

Estimates (4.23) and (4.22), the assumptions on $\{M_n\}$, and an induction on n guarantee that the estimate

$$(4.24) \quad \left\| \frac{\partial^n}{\partial t^n} u(t,\lambda) \right\|_\omega + |\lambda| \left\| \frac{\partial^n}{\partial t^n} u(t,\lambda) \right\|_0 \leqq H_0 H^n M_n (1 + |\lambda|^{1/\omega})^{-1} \| f \|_0$$

holds for $n = 0, 1, \cdots$ provided that $H > \max(2B, 4d_1 B_0 B)$.

Let $0 \leqq \theta_1 < \theta_2 < 2\pi$ and suppose that $\theta_2 - \theta_1 \leqq \pi$. Assume that $A(t,x,D)$ and $B_j(t,x,D)$, $j = 1, \cdots, \frac{1}{2}\omega l$, satisfy Agmon's conditions on the ray l_{θ_i} for $i = 1, 2$. Suppose that for $i = 1, 2$, $l_{\theta_i} \subset \rho(A_B^p(t))$. The results of [3] ensure the existence of a bounded projection $E_p(t, \theta_1, \theta_2)$ in $H^{0,p}(G)$ that satisfies the following requirements: for $f \in D(A_B(t))$ we have $E_p(t,\theta_1,\theta_2) f \in D(A_B(t))$ and $A_B(t) E_p(t,\theta_1,\theta_2) f = E_p(t,\theta_1,\theta_2) A_B^p(t) f$. Also $\sigma(A_B^p(t) E_p(t,\theta_1,\theta_2)) - \{0\} = \sigma(A_B^p(t)) \cap \Gamma(\theta_1,\theta_2)$ and there exists a constant $c(t)$ such that for $\lambda \notin \Gamma(\theta_1, \theta_2)$ we have

$$\| (\lambda - A_B^p(t) E_p(t,\theta_1,\theta_2))^{-1} \| \leqq c(t) \, |\lambda|^{-1}.$$

We remark that arguments similar to those used in the proof of part (ii) of

Lemma 4.7 below may be used to extend the above-mentioned result to the case $\theta_1 - \theta_2 > \pi$. Note that in case $\pi/2 < \theta_1 < \theta_2 < 3\pi/2$ we have

$$E_p(t,\theta_1,\theta_2)f = \lim_{\tau\to 0} \int_{\gamma(\theta_1,\theta_2)} e^{\lambda\tau}(\lambda - A_B^p(t))^{-1} d\lambda f \tag{4.25}$$

for every $f \in H^{0,p}(G)$. (See [3].)

LEMMA 4.7. *Let $\pi/2 < \theta_1 < \theta_2 < 3\pi/2$. Suppose that the assumptions of Lemma 4.2 are satisfied for $\theta = \theta_i$, $i = 1,2$. Assume that for $t \in [a,b]$ and $i = 1,2$, $l_{\theta_i} \subset \rho(A_B^p(t))$. For $t \in [a,b]$ set $E_p(t) = E_p(t,\theta_1,\theta_2)$.*

(i) *There exist constants H_0 and H such that*

$$E_p(t) \in G(H_0, H, [a,b], B(H^{0,p}(G), H^{0,p}(G))).$$

(ii) *There exists a complex μ_0 such that conditions* (i), (ii), *and* (iii)′ *of Section* 3 *are satisfied by $L(t) = A_B^p(t) + \mu_0 I$.*

PROOF. Part (i) is an immediate consequence of (4.25), of Lemma 4.4 (ii), and of Lemma 4.6.

It follows from the assumptions of the present lemma that there exists a $\delta > 0$ such that for $t \in [a,b]$ we have $\Gamma(\theta_1 - \delta, \theta_1) \cup \Gamma(\theta_2, \theta_2 + \delta) \subset \rho(A_B^p(t))$. Let

$$\mu \in \Gamma(\theta_1 - \delta, \theta_1) \cup \Gamma(\theta_2, \theta_2 + \delta) - \gamma(\theta_1, \theta_2).$$

Then

$$(\mu - A_B^p(t)E_p(t))^{-1} = (\mu - A_B^p(t))^{-1}E_p(t) + \mu^{-1}(I - E_p(t)).$$

The resolvent equation,

$$(\mu - A_B^p(t)^{-1}(\lambda - A_B^p(t))^{-1} = (\mu - \lambda)^{-1}((\lambda - A_B^p(t))^{-1} - (\mu - A_B^p(t))^{-1})$$

combined with (4.25) implies that

$$(\mu - A_B^p(t))^{-1}E_p(t) = \frac{1}{2\pi i}\int_{\gamma(\theta_1,\theta_2)} (\lambda - A_B^p(t))^{-1}(\mu - \lambda)^{-1} d\lambda.$$

Consequently for $\mu \in \Gamma(\theta_1 - \delta, \theta_1) \cup \Gamma(\theta_2, \theta_2 + \delta) - \gamma(\theta_1, \theta_2)$ we have

$$(\mu - A_B^p(t)E_p(t))^{-1} = \frac{1}{2\pi i}\int_{\gamma(\theta_1,\theta_2)} (\lambda - A_B^p(t))^{-1}(\mu - \lambda)^{-1} d\lambda + \mu^{-1}(I - E_p(t)). \tag{4.26}$$

The right-hand side of (4.26) is analytic in the complement of $\gamma(\theta_1,\theta_2)$ and since $\sigma(A_B^p(t)E_p(t)) \subset \Gamma(\theta_1,\theta_2)$, the relation (4.26) holds for $\mu \notin \Gamma(\theta_1,\theta_2)$. The validity of (4.26) for $\mu \notin \Gamma(\theta_1,\theta_2)$, Lemma 4.4 (iii), Lemma 4.6, and (i) of the present lemma ensure that for every sufficiently small $\varepsilon > 0$ there exist constants H_0 and H such that for $\mu \notin \Gamma(\theta_1 - \varepsilon, \theta_2 + \varepsilon)$ we have

$$(\mu - A_B^p(t)E_p(t))^{-1} \in G(|\mu|^{-1}(H_0, H, [a,b], B(H^{0,p}(G), H^{0\,p}(G))))$$

and consequently the assertion of part (ii) is true.

THEOREM 4.8. *Let $\{M_n\}$ be a sequence of positive constants that satisfy the requirements (2.1) through (2.4). Denote by $a(t,x)$ any of the coefficients of $A(t,x,D)$ or of $B_j(t,x,D)$, $j = 1,\cdots,\frac{1}{2}\omega l$. Assume that $a(t,x)$ is infinitely differentiable in $[a,b] \times \bar{G}$ and that for every multi-index α there exist constants H_0 and H such that for $n = 0,1 \cdots$ and $(t,x) \in [a,b] \times \bar{G}$ we have*

$$\left| \frac{\partial^n}{\partial t^n} \frac{\partial_\alpha}{\partial x_\alpha} a(t,x) \right| \leqq H_0 H^n M_n.$$

Assume that for $t \in [a,b]$, $A(t,x,D)$ and $B_j(t,x,D), j = 1,\cdots,\frac{1}{2}\omega l$, satisfy Agmon's conditions on the rays $l_{\frac{1}{2}\pi}$ and $l_{-\frac{1}{2}\pi}$. Suppose that $f(t) \in G(F_0, F, [a,b])$. Let $u(t) \in C[a,b] \cap C^1(a,b)$. Suppose that for $t \in (a,b)$ we have $u(t) \in D(A_B^p(t))$ and

$$\frac{du}{dt} - A_B^p(t)u(t) = f(t). \tag{4.27}$$

Then $u(t) \in C^\infty(a,b)$ and there exist constants C and B such that for $n = 1,2,\cdots$ and $t \in (a,b)$ we have

$$\begin{aligned} \left\| \frac{d^n}{dt^n} u(t) \right\| \leqq\ & B^n M_n \Big((t-a)^{-n} \| E_1(a)u(a) \| + (b-t)^{-n} \| E_2(b)u(b) \| \\ & + (t-a)^{-n+1}(b-t)^{-n+1}\big(\max_{t \in [a,b]} \| u(t) \| + F_0 \big) \Big). \end{aligned} \tag{4.28}$$

$E_1(a)$ and $E_2(b)$ are bounded projections in $H^{0,p}(G)$ such that $A_B^p(a)E_1(a)$ and $-A_B^p(b)E_2(b)$ are infinitesimal generators of analytic semigroups.

PROOF. To prove Theorem 4.8 it is sufficient to verify that for every $t_0 \in [a,b]$ there exists an $r > 0$ such that $A_B^p(t)$ satisfies the requirements of Theorem 3.6 in the interval $[a,b] \cap \{t; |t - t_0| \leqq r\}$. Let $t_0 \in [a,b]$ and suppose that

$$0 \in \rho(A_B^p(t_0)).$$

The assumptions of the present theorem guarantee the existence of a $\delta > 0$ such that for $t \in [a,b]$ and $\theta \in [-\frac{1}{2}\pi - \delta, -\frac{1}{2}\pi + \delta] \cup [\frac{1}{2}\pi - \delta, \frac{1}{2}\pi + \delta]$, $A(t,x,D)$ and $B_j(t,x,D)$ $j = 1, \cdots \frac{1}{2}\omega l$, satisfy Agmon's conditions on l_θ. Choose R so that for $t \in [a,b]$ and $\lambda \in \delta(\frac{1}{2}\pi - \delta, \frac{1}{2}\pi + \delta) \cup (\frac{1}{2}\pi - \delta, \frac{1}{2}\pi + \delta)$ with $|\lambda| \geqq R$ we have $\lambda \in \rho(A_B^p(t_0))$ and estimate (4.14) holds. Noting the discreteness of the spectrum of $A_B^p(t_0)$, choose δ so that in addition to the above-mentioned assumptions the rays $l - \frac{1}{2}\pi - \delta$, $l - \frac{1}{2}\pi + \delta$, $\frac{1}{2}\pi l - \delta$, and $\frac{1}{2}\pi l + \delta$ belong to $\rho(A_B(t_0))$. Let ∂ be the boundary of $\{\lambda; |\lambda| \leqq R, \lambda \in \Gamma(-\frac{1}{2}\pi - \delta, -\frac{1}{2}\pi + \delta) \cup \Gamma(\frac{1}{2}\pi - \delta, \frac{1}{2}\pi + \delta)\}$. The relation $\partial \subset \rho(A_B^p(t_0))$, the compactness of ∂ and the validity of (4.14) for some λ ensure the existence of positive constants r, H_0 and H such that for $t \in [a, b]$ with $|t - t_0| \leqq r$ and $\lambda \in \partial$ we have $\lambda \in \rho(A_B^p(t))$ and

$$\left\| \frac{\partial^n}{\partial t^n}(\lambda - A_B^p(t))^{-1} \right\| \leqq H_0 H^n M_n. \tag{4.29}$$

Set $[\alpha, \beta] = [a, b] \cap \{t; |t - t_0| \leqq r\}$ and for $t \in [\alpha, \beta]$ let

$$E_p^0(t) = \frac{1}{2\pi i} \int_\partial (\lambda - A_B^p(t))^{-1} d\lambda.$$

It follows from estimate (4.29) that there exist constants H_0 and H such that

$$E_p^0(t) \in G(H_0, H, [\alpha, \beta], B(H^{0,p}(G), H^{0,p}(G)))$$

and

$$A_B^p(t) E_p^0(t) \in G(H_0, H, [\alpha, \beta], B(H^{0,p}(G), H^{0,p}(G))).$$

For $t \in [\alpha, \beta]$, let $E_1(t) = E_p(t, \frac{1}{2}\pi + \delta, 3\pi/2 - \delta) + E_p^0(t)$ and set $E_2(t) = E_p(t, -\frac{1}{2}\pi + \delta, \frac{1}{2}\pi - \delta)$. The results of [3] guarantee that for $t \in [\alpha, \beta]$, $A_B^p(t)$ is completely reduced by the direct sum decomposition

$$H^{0,p}(G) = \sum_{i=1}^{2} \oplus E_i(t) H^{0,p}(G).$$

Consequently Lemma 4.7 and the above-mentioned properties of $E_p^0(t)$ and $A_B^p(t)E_p^0(t)$ guarantee that $A_B^p(t)$ satisfies the assumptions of Theorem 3.6 in $[\alpha, \beta]$. In case $t_0 \in [a, b]$ and $0 \notin \rho(A_B^p(t_0))$, a result of the same type is obtained by considering the family $A_B^p(t) - \lambda_0 I$ where $\lambda_0 \in \rho(A_B^p(t_0))$.

Note that Theorem 4.8 extends the results of [9] where it is assumed that for each $t \in [a, b]$, $A_B^p(t)$ is the infinitesimal generator of an analytic semigroup. In this case there exist constants H_0, H, G_0, and G such that for $n \geqq 1$ and $t \in (a, b]$ we have

$$\left\| \frac{d^n}{dt^n} u(t) \right\| \leqq H_0 H^n M_n (t - a)^{-n} \| u(a) \| + G_0 G^n M_n (t - a)^{-n+1}. \tag{4.30}$$

See [9].

We state without proof the Theorem 4.9 that can be proved with the help of Theorem 3.3. by a method similar to that used in Theorem 4.8.

THEOREM 4.9. *Assume that the coefficients of* $A(t, x, D)$ *and* $B_j(t, x, D)$, $j = 1, \cdots, \frac{1}{2}\omega l$, *are infinitely differentiable in* $[a, b] \times \bar{G}$. *Suppose that for*

$t \in [a,b]$, $A(t,x,D)$ and $B_j(t,x,D)$, $j = 1,\cdots,\frac{1}{2}\omega l$, satisfy Agmon's conditions on the rays $l_{\frac{1}{2}\pi}$ and $l_{-\frac{1}{2}\pi}$. Let $u(t) \in C[a,b] \cap C^1(a,b)$, and suppose that for $t \in (a,b)$, $u(t) \in D(A_B^p(t))$ and $du/dt - A_B^p(t)u(t) = f(t)$. Then $u(t) \in C^\infty(a,b)$ and for every positive integer n there exists a constant C_n such that for $t \in (a,b)$ and $n = 1,2,\cdots$ we have

$$\left\| \frac{d^n}{dt^n} u(t) \right\| \leqq C_n((t-a)^{-n} \| E_1(a)u(a) \| + (b-t)^{-n} \| E_2(b)u(b) \| + (t-a)^{-n+1}(b-t)^{-n+1}\left(\max_{t \in [a,b]} \| u(t) \| + \max_{\substack{k=0,\cdots,n \\ t \in [a,b]}} \left\| \frac{d^k}{dt^k} f(t) \right\| \right). \tag{4.31}$$

$E_1(a)$ and $E_2(b)$ are bounded projections in $H^{0,p}(G)$ such that $A_B^p(a)E_1(a)$ and $-A_B^p(b)E(b)$ are infinitesimal generators of analytic semigroups.

Note that for $p = 2$, or for $1 < p < \infty$ and $A_B^p(t)$ independent of t, the assumptions of Theorem 4.9 and the results of [10] and [2] respectively guarantee that $u(t) \in C^\infty(a,b)$.

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